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Operators for One-Sided Approximation by Algebraical Polynomials*

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1. INTRODUCTION

In this paper we construct and analyse one-sided approximation operators for functions of bounded variation on [0, 1]. The basic idea of the construction consists in connecting the classical approach for one-sided approximation used by Freud [1] and Névai [5] with a special problem of Hermite interpolation. To get characterization and saturation for one-sided L_p-approximation, we will use the well known τ-modulus which is the appropriate modulus for measuring smoothness of functions in one-sided approximation theory (cf. Popov [6] and his references). This paper includes also some brief remarks concerning the weighted L_p-approximation properties of the operators. In this context, smoothness of a function f will be measured by the properties of some special maximal functions of f.

2. NOTATION

Let $BV[0, 1]$ be the space of all functions f of bounded variation on $[0, 1]$ canonically extended to $\mathbb{R}$ by defining $f(x) := f(0)$, $x < 0$, and $f(x) := f(1)$, $x > 1$. For $1 \leqslant p < \infty$ we will denote by $L_p[0, 1]$ the class of measurable functions f on $[0, 1]$ with $|f|^p$ Lebesgue integrable and by $\|\cdot\|_p$ the usual L_p-norm with respect to $[0, 1]$. Finally, let $C[0, 1]$ be the space of all continuous functions on $[0, 1]$ and $\|\cdot\|_\infty$ the corresponding maximum norm.

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3. Construction of the Operators

We apply the classical strategy of generating one-sided approximations used by Freud [1] and Névai [5]. So we are first interested in constructing simple but effective one-sided approximations for the so-called step functions $G_0: \mathbb{R}^2 \to \{0, 1\}$,

$$G_0(x, t) := (x-t)_+^0 := \begin{cases} 0 & \text{if } x < t \\ 1 & \text{if } x \geqslant t. \end{cases}$$

For this purpose we generate uniquely determined Hermite interpolation polynomials $h_n(x, t)$ and $H_n(x, t)$ of maximal degree $2n$ in x which are defined by

$$\begin{aligned} h_n^{(i)}(t-1, t) = H_n^{(i)}(t-1, t) &= 0, \qquad i = 0, ..., n-1, \\ h_n(t, t) &= 0, \\ H_n(t, t) &= 1, \\ h_n(t+1, t) = H_n(t+1, t) &= 1, \\ h_n^{(i)}(t+1, t) = H_n^{(i)}(t+1, t) &= 0, \qquad i = 1, ..., n-1. \end{aligned} \tag{3.1}$$

(It is assumed that $t \in \mathbb{R}$ is fixed and the differentiation is taken with respect to x. For sake of brevity we will sometimes identify a function f with its "value" $f(x)$, as done above in case of h_n and H_n.)

Lemma 3.1. *Let $n \in \mathbb{N}$ and $t \in [0, 1]$ be given. Then the algebraic polynomials $h_n(x, t)$ and $H_n(x, t)$ of maximal degree $2n$ in x satisfy the inequality*

$$h_n(x, t) \leqslant (x-t)_+^0 \leqslant H_n(x, t), \qquad x \in [0, 1], \tag{3.2}$$

and the identity

$$(H_n(x, t) - h_n(x, t)) = (1-(x-t)^2)^n, \qquad x \in [0, 1]. \tag{3.3}$$

Moreover, for a fixed $x \in [0, 1]$, $h_n(x, t)$ and $H_n(x, t)$ are continuous functions in t.

Proof. The lemma may be easily proved by explicit calculation of the functions h_n and H_n. We get

$$h_n(x, t) = a_n \int_{t-1}^{x} (1-(\xi-t)^2)^{n-1}\, d\xi - \tfrac{1}{2}(1-(x-t)^2)^n,$$

$$H_n(x, t) = a_n \int_{t-1}^{x} (1-(\xi-t)^2)^{n-1}\, d\xi + \tfrac{1}{2}(1-(x-t)^2)^n,$$

with

$$a_n := \left\{\int_{t-1}^{t+1} (1-(\xi-t)^2)^{n-1}\, d\xi\right\}^{-1} = \left\{\int_{-1}^{1} (1-\xi^2)^{n-1}\, d\xi\right\}^{-1}$$

To get (3.2) we additionally have to show that $h_n(x, t)$ (and analogously $H_n(x, t)$) has only one local extremum in $(t-1, t+1) \supset (0, 1)$ as a function of x. This follows immediately by the equivalence

$$h_n'(x, t) = 0 \qquad \text{for} \quad x \in (t-1, t+1)$$

if and only if $x = t - a_n n^{-1}$.

To do the final step in constructing the operators we need a few definitions. For $f \in BV[0, 1]$ and $0 \leqslant x < y \leqslant 1$ let $V_x^y(f)$ be the total variation of f over $[x, y]$ and $V_f(x) := V_0^x(f)$, $x \in [0, 1]$, the so-called variation function of f. Moreover, we define

$$f^+(x) := \tfrac{1}{2}(V_f(x) + f(x))$$

and

$$f^-(x) := \tfrac{1}{2}(V_f(x) - f(x))$$

for $x \in [0, 1]$. As f^+ and f^- are non-decreasing functions we get the following theorem.

THEOREM 3.1. *Let $n \in \mathbb{N}$ and $f \in BV[0, 1]$ be given. Then the operators φ_n and Φ_n defined on $BV[0, 1]$ by*

$$\begin{aligned} \varphi_n(f)(x) &:= f(0) + \int_0^1 h_n(x, t)\, df^+(t) - \int_0^1 H_n(x, t)\, df^-(t), \\ \Phi_n(f)(x) &:= f(0) + \int_0^1 H_n(x, t)\, df^+(t) - \int_0^1 h_n(x, t)\, df^-(t), \end{aligned} \tag{3.4}$$

have the following properties:

(a) $\varphi_n(f), \Phi_n(f) \in \Pi_{2n}$,

(b)

$$\varphi_n(f)(x) \leqslant f(x) \leqslant \Phi_n(f)(x), \qquad x \in [0, 1], \tag{3.5}$$

(c) *For $x \in [0, 1]$ the one-sided approximation error given by φ_n and Φ_n has the representation*

$$(\Phi_n(f)(x) - \varphi_n(f)(x)) = \int_0^1 (1-(x-t)^2)^n\, dV_f(t). \tag{3.6}$$

Proof. Since (a) and (c) are easy consequences of Lemma 3.1 and the linearity of the Riemann–Stieltjes integral (both in integrand and integrator) we only have to prove (b). Because of Lemma 3.1 we get using the monotonicity of the Riemann–Stieltjes integral in case of monotone integrator functions ($x \in [0, 1]$),

$$\begin{aligned} f(x) &= f(0) + \int_0^x df^+(t) - \int_0^x df^-(t) \\ &\geqslant f(0) + \int_0^x h_n(x, t)\, df^+(t) - \int_0^x H_n(x, t)\, df^-(t) \\ &\geqslant f(0) + \int_0^1 h_n(x, t)\, df^+(t) - \int_0^1 H_n(x, t)\, df^-(t) \\ &= \varphi_n(f)(x). \end{aligned}$$

Analogously we can show $f(x) \leqslant \Phi_n(f)(x)$, $x \in [0, 1]$.

Remarks. (1) The operators φ_n and Φ_n are non-linear since the only linear one-sided approximation operator is the identity operator.

(2) It can be easily shown that the operators are equicontinuous in the sense of Hölder as functions from the Banach space $BV[0, 1]$ with the variation norm into $C[0, 1]$ with the maximum norm. For if we define the variation norm by $\|q\|_v := |q(0)| + V_0^1(q)$, $q \in BV[0, 1]$, and choose $f, g \in BV[0, 1]$ arbitrarily we get by means of the inequalities $|h_n(x, t)| \leqslant 2$ and $|H_n(x, t)| \leqslant 2$, $x, t \in [0, 1]$,

$$\begin{aligned} \|\varphi_n(f) - \varphi_n(g)\|_\infty &\leqslant |f(0) - g(0)| + 2V_0^1(f^+ - g^+) + 2V_0^1(f^- - g^-) \\ &\leqslant |(f-g)(0)| + 2V_0^1(f-g) + 2V_0^1(V_f - V_g) \\ &\leqslant 4\, \|f-g\|_v \end{aligned}$$

and analogously

$$\|\Phi_n(f) - \Phi_n(g)\|_\infty \leqslant 4\, \|f-g\|_v.$$

4. Local and Global Approximation Properties

To get a first impression of the approximation properties of the operators we start with a local approximation theorem. For this reason we define $f(x+) := \lim_{h \to 0+} f(x+h)$ and $f(x-) := \lim_{h \to 0+} f(x-h)$ and remember that for $f \in BV[0, 1]$ and $x \in [0, 1]$ we have $V_{x-}^{x+}(f) = |f(x+) - f(x)| + |f(x) - f(x-)|$ (cf. Riesz/Sz.-Nagy [7, p. 14]).

THEOREM 4.1. *Let $f \in BV[0, 1]$ be given. Then we have for $x \in [0, 1]$,*

$$\lim_{n \to \infty} (\Phi_n(f) - \varphi_n(f))(x) = |f(x+) - f(x)| + |f(x) - f(x-)|. \tag{4.1}$$

Moreover, if f is non-increasing or non-decreasing at x, i.e., if

$$\min\{f(x-), f(x+)\} \leqslant f(x) \leqslant \max\{f(x-), f(x+)\},$$

we have

$$\begin{aligned} \lim_{n \to \infty} \varphi_n(f)(x) &= \min\{f(x-), f(x+)\}, \\ \lim_{n \to \infty} \Phi_n(f)(x) &= \max\{f(x-), f(x+)\}. \end{aligned} \tag{4.2}$$

Proof. Since (4.2) follows immediately from (3.5), (4.1), and the local monotonicity of f, we only have to prove (4.1). Using the inequality $(1-(x-t)^2)^n \leqslant e^{-n(x-t)^2}$, $x, t \in [0, 1]$, and the assumed constant extension of f over $[0, 1]$ we get the following estimates:

$$\begin{aligned} \Phi_n(f)(x) - \varphi_n(f)(x) &\leqslant \int_0^1 e^{-n(x-t)^2}\, dV_f(t) \\ &\leqslant e^{-\sqrt{n}} V_0^1(f) + V_{x-n^{-1/4}}^{x+n^{-1/4}}(f), \\ \Phi_n(f)(x) - \varphi_n(f)(x) &\geqslant \int_{x-n^{-1}}^{x+n^{-1}} (1-(x-t)^2)^n\, dV_f(t) \\ &\geqslant \left(1 - \frac{1}{n^2}\right)^n V_{x-n^{-1}}^{x+n^{-1}}(f). \end{aligned}$$

Now (4.1) follows for $n \to \infty$.

After this local result we are now interested in the more global approximation properties of the operators. For this reason we have to introduce the well known τ-modulus of first order of f with respect to p. Using the local ω-modulus (f bounded and measurable, $x \in [0, 1]$, $\delta > 0$)

$$\omega_1(f, x, \delta) := \sup\left\{|f(a) - f(b)| : a, b \in \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right] \cap [0, 1]\right\},$$

it is defined by

$$\begin{aligned} \tau_{1,p}(f, \delta) &:= \|\omega_1(f, x, \delta)\|_p, \qquad 1 \leqslant p < \infty, \\ \tau_{1,\infty}(f, \delta) &:= \sup\{\omega_1(f, x, \delta) : x \in [0, 1]\} = \omega(f, \delta). \end{aligned}$$

LEMMA 4.1. *Let $f \in BV[0, 1]$ and $1 \leqslant p \leqslant \infty$ be given. Then we have*

$$\text{(a)} \quad \tau_{1,p}(f, \delta) \leqslant \tau_{1,p}(f, \delta'), \qquad 0 < \delta \leqslant \delta', \tag{4.3}$$

$$\text{(b)} \quad \tau_{1,p}(f, n\delta) \leqslant n\tau_{1,p}(f, \delta), \qquad 0 < \delta, n \in \mathbb{N}, \tag{4.4}$$

(c) *f is constant on $[0, 1]$ if and only if*

$$\tau_{1,p}(f, \delta) = o(\delta) \qquad (\delta \to 0). \tag{4.5}$$

Proof. For (a) and (b) see Popov [6] and his references. Moreover, for $p = \infty$ equivalence (c) is well known, too (for example see Görlich/Nessel [2, Remark 4.3]). Since the τ-modulus vanishes for constant functions one implication of (c) is evident even in case $1 \leqslant p < \infty$. To get the opposite implication we remember the inequality $\omega_{1,p}(f, \delta) \leqslant \tau_{1,p}(f, \delta)$ where $\omega_{1,p}(f, \delta)$ is the usual ω-modulus of first order with respect to L_p (cf. Popov [6]). Now the small o-condition for the τ-modulus gives $\omega_{1,p}(f, \delta) = o(\delta)$, $\delta \to 0$, which implies that f is constant almost everywhere on $[0, 1]$ (see again Görlich/Nessel [2]). The final step of showing that f is constant everywhere on $[0, 1]$ is easily done by contradiction ($f = C$ a.e. on $[0, 1]$ and $f(x_0) = C_0 \neq C$ for some $x_0 \in [0, 1]$ implies $\tau_{1,p}(f, \delta) \geqslant M\delta$, $M > 0$; for details see [3, 4]).

We are now able to formulate a complete characterization and saturation theorem for the operators φ_n and Φ_n.

THEOREM 4.2. *Let $f \in BV[0, 1]$, $1 \leqslant p \leqslant \infty$, and $n \in \mathbb{N}$, $n \geqslant 2$, be given. Then there exist non-negative real numbers K_1 and K_2 (independent of f, p, and n), such that*

$$\text{(i)} \quad \tau_{1,p}(V_f, n^{-1/2}) \leqslant K_1 \, \|\Phi_n(f) - \varphi_n(f)\|_p, \tag{4.6}$$

$$\text{(ii)} \quad \|\Phi_n(f) - \varphi_n(f)\|_p \leqslant K_2 \tau_{1,p}(V_f, n^{-1/2}). \tag{4.7}$$

In particular we have for $0 < \alpha \leqslant \frac{1}{2}$,

$$\|\Phi_n(f) - \varphi_n(f)\|_p = O(n^{-\alpha}) \qquad (n \to \infty),$$

if and only if

$$\tau_{1,p}(V_f, \delta) = O(\delta^{2\alpha}) \qquad (\delta \to 0).$$

Finally the operators are saturated of order $n^{-1/2}$, i.e.,

$$\|\Phi_n(f) - \varphi_n(f)\|_p = O(n^{-1/2}) \qquad (n \to \infty)$$

if and only if

$$\tau_{1,p}(V_f, \delta) = O(\delta) \qquad (\delta \to 0)$$

and

$$\|\Phi_n(f) - \varphi_n(f)\|_p = o(n^{-1/2}) \qquad (n \to \infty)$$

if and only if f is constant on $[0, 1]$.

Proof. Since with Lemma 4.1 the characterization and saturation statements follow easily from (i) and (ii) by standard analysis, we only have to prove the two inequalities.

(i) Since $(1-1/n)^n \geqslant 1/2e$, $n \geqslant 2$, we have for $1 \leqslant p < \infty$

$$\begin{aligned}
\tau_{1,p}(V_f, n^{-1/2}) &= \left(\int_0^1 \{V_{x-(1/2)n^{-1/2}}^{x+(1/2)n^{-1/2}}(f)\}^p\, dx\right)^{1/p}\\
&\leqslant \left(\int_0^1 \left\{\int_{|x-t|\leqslant n^{-1/2}} 2e\left(1-\frac{1}{n}\right)^n dV_f(t)\right\}^p dx\right)^{1/p}\\
&\leqslant 2e\left(\int_0^1 \left\{\int_0^1 (1-(x-t)^2)^n\, dV_f(t)\right\}^p dx\right)^{1/p}\\
&= 2e\, \|\Phi_n(f) - \varphi_n(f)\|_p.
\end{aligned}$$

(ii) Let $x \in [0, 1]$ be given and $1 \leqslant p < \infty$. Defining $[\sqrt{n}] := \max\{m \in \mathbb{Z} \mid m \leqslant \sqrt{n}\}$ we introduce the equidistant partition $0 = t_0 < t_1 < \cdots < t_{[\sqrt{n}]+1} = 1$ of $[0, 1]$ with $t_i := i \cdot ([\sqrt{n}]+1)^{-1}$, $i = 0, ..., [\sqrt{n}]+1$. Assuming $x \in [t_k, t_{k+1}]$ we get, by using the so-called upper Riemann–Stieltjes sum with respect to the chosen partition,

$$\begin{aligned}
\Phi_n(f)(x) - \varphi_n(f)(x) &\leqslant \int_0^1 e^{-n(x-t)^2}\, dV_f(t)\\
&\leqslant \sum_{i=0}^{k-1} e^{-n(x-t_{i+1})^2} V_{t_i}^{t_{i+1}}(f)\\
&\quad + 1 \cdot V_{t_k}^{t_{k+1}}(f)\\
&\quad + \sum_{i=k+1}^{[\sqrt{n}]} e^{-n(x-t_i)^2} V_{t_i}^{t_{i+1}}(f)\\
&\leqslant \sum_{i=0}^{[\sqrt{n}]} e^{-n(i/([\sqrt{n}]+1))^2} \omega_1\left(V_f, x, \frac{2(i+2)}{[\sqrt{n}]+1}\right).
\end{aligned}$$

Taking the L_p-norm on both sides we finally get, by means of Lemma 4.1,

$$\|\Phi_n(f)-\varphi_n(f)\|_p \leqslant \sum_{i=0}^{[\sqrt{n}]} e^{-i^2/4}\tau_{1,p}\left(V_f, \frac{2(i+2)}{[\sqrt{n}]+1}\right)$$
$$\leqslant \left(\sum_{i=0}^{\infty} 2(i+2)\, e^{-i^2/4}\right)\tau_{1,p}(V_f, n^{-1/2}).$$

The proof for $p=\infty$ follows analogously by taking the supremum instead of the integrals.

Remark. For $f\in BV[0,1]$ the saturation condition $\tau_{1,p}(V_f,\delta)=O(\delta)$, $(\delta\to 0)$, is equivalent to $\tau_{1,p}(f,\delta)=O(\delta)$, $(\delta\to 0)$. For a detailed discussion of the connection between the τ-modulus of f and the τ-modulus of the variation function of f see [4].

5. Weighted L_p-Approximation Properties

Let $BV(\mathbb{R})$ be the space of all real valued functions of bounded variation on $\mathbb{R}$. For $f\in BV(\mathbb{R})$ and $0\leqslant\alpha\leqslant 1$ let f_α^{**} denote the modified maximal function of f,

$$f_\alpha^{**}(x) := \sup_{h>0} \frac{|f(x+h)-f(x-h)|}{(2h)^\alpha}, \qquad x\in\mathbb{R}. \tag{5.1}$$

Since in this paper we only consider functions $f\in BV(\mathbb{R})$ the modified maximal functions may be interpreted as a special case of the maximal functions mentioned by Stein and Weiss [8, p. 85, Remark 5.6] and others especially in connection with relative differentiation of measures; to see this we note that each $f\in BV(\mathbb{R})$ induces a well-defined Borel measure, the so-called Lebesgue–Stieltjes measure corresponding to f. Moreover, $f_\alpha^{**}(x)$ can be interpreted as a local Lipschitz constant of f of order α. Since f is of bounded variation the central derivates of f exist almost everywhere on $\mathbb{R}$, i.e., $f_\alpha^{**}(x)$ is finite for almost every $x\in\mathbb{R}$. More precisely, $f_\alpha^{**}(x)$ is finite if and only if $|f(x+h)-f(x-h)|=O((2h)^\alpha)$, $h\to 0$, i.e., the modified maximal functions are sensitive measures of the local divergence or convergence order of the central derivates of f.

Lemma 5.1. *Let $f\in BV(\mathbb{R})$ be non-decreasing and $0\leqslant\alpha\leqslant 1$. Then there exists a real constant $C>0$ (independent of f and α) such that for all $x\in\mathbb{R}$*

$$\sup_{n\in\mathbb{N}} \int_{-\infty}^{\infty} n^{\alpha/2} e^{-n(x-t)^2}\, df(t) \leqslant C\cdot f_\alpha^{**}(x). \tag{5.2}$$

Proof. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$ be given and define $t_i := x + in^{-1/2}$, $i \in \mathbb{Z}$. Then we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} n^{\alpha/2} e^{-n(x-t)^2}\, df(t) \\
&\quad \leqslant \limsup_{\substack{i \to \infty \\ i \in \mathbb{N}}} \sum_{j=0}^{i-1} n^{\alpha/2} e^{-j^2} (f(t_{j+1}) - f(t_j)) \\
&\qquad + \limsup_{\substack{i \to -\infty \\ -i \in \mathbb{N}}} \sum_{j=i+1}^{0} n^{\alpha/2} e^{-j^2} (f(t_j) - f(t_{j-1})) \\
&\quad \leqslant \lim_{\substack{i \to \infty \\ i \in \mathbb{N}}} \sum_{j=0}^{i-1} e^{-j^2} (2(j+1))^{\alpha} \\
&\qquad \times \frac{f(x + (j+1)/\sqrt{n}) - f(x - (j+1)/\sqrt{n})}{(2(j+1)/\sqrt{n})^{\alpha}} \\
&\qquad + \lim_{\substack{i \to -\infty \\ -i \in \mathbb{N}}} \sum_{j=i+1}^{0} e^{-j^2} (2(|j|+1))^{\alpha} \\
&\qquad \times \frac{f(x + (|j|+1)/\sqrt{n}) - f(x - (|j|+1)/\sqrt{n})}{(2(|j|+1)/\sqrt{n})^{\alpha}} \\
&\quad \leqslant \left(4 \cdot \sum_{j=0}^{\infty} e^{-j^2} (j+1)\right) f_{\alpha}^{**}(x).
\end{aligned}$$

Remark. In case $\alpha = 1$ it follows from Lemma 5.1 that

$$\sup_{n \in \mathbb{N}} \left| \int_{-\infty}^{\infty} \sqrt{n}\, e^{-n(x-t)^2} g(t)\, dt \right| \leqslant C \cdot g^*(x)$$

for $g \in L_1(\mathbb{R})$, $x \in \mathbb{R}$, and g^* the classical Hardy–Littlewood maximal function of g,

$$g^*(x) := \sup_{h > 0} \frac{1}{2h} \int_{x-h}^{x+h} |g(t)|\, dt$$

(cf. Wheeden/Zygmund [9, pp. 104, 156]). Moreover, an easy calculation yields

$$f_1^{**}(x) \leqslant (f')^*(x), \qquad x \in \mathbb{R},$$

in case of f being absolutely continuous on $\mathbb{R}$. This inequality shows the essential difference between the two maximal functions: while the Hardy–

Littlewood maximal function of f is a gauge of the size of the averages of $|f|$ around x, the modified maximal function is a measure of the size of the averages of the absolute derivates of f around x.

Now we are able to give a sufficient condition for a prescribed weighted approximation order in terms of the integration properties of the modified maximal functions.

THEOREM 5.2. *Let $f \in BV[0, 1]$, $1 \leqslant p < \infty$, and $0 \leqslant \alpha \leqslant 1$ be given. Moreover, let $g \in L_p[0, 1]$ be a non-negative weight function. If the integral*

$$\int_0^1 ((V_f)_\alpha^{**}(x)\, g(x))^p\, dx$$

exists and is finite, then we have for $n \to \infty$,

$$\|(\Phi_n(f) - \varphi_n(f))\, g\|_p = O(n^{-\alpha/2}).$$

Proof. For given $n \in \mathbb{N}$ we get using Lemma 5.1

$$\begin{aligned} & n^{\alpha/2}\, \|(\Phi_n(f) - \varphi_n(f))\, g\|_p \\ & \leqslant \left(\int_0^1 \left(\int_0^1 n^{\alpha/2} e^{-n(x-t)^2}\, dV_f(t) \right)^p (g(x))^p\, dx \right)^{1/p} \\ & \leqslant C \cdot \|(V_f)_\alpha^{**} \cdot g\|_p. \end{aligned}$$

(Note that in this theorem f and V_f are assumed to be extended on $\mathbb{R}$ constantly.)

Remark. Let us mention that by means of the proofs of Lemma 5.1 and Theorem 5.2 the condition

$$\|(V_f(\cdot + h) - V_f(\cdot - h))\, g\|_p = O(|h|^\alpha), \qquad h \to 0,$$

is also sufficient for the validity of Theorem 5.2. The concept of maximal functions, however, is more directly connected with the smoothness of f resp. V_f and, for example, also works in case $0 < p < 1$.

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